

STATISTICAL EXPONENTIAL FORMULAS FOR HOMOGENEOUS DIFFUSION

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ABSTRACT. Let Δ_p^1 denote the 1-homogeneous p -Laplacian, for $1 \leq p \leq \infty$. This paper proves that the unique bounded, continuous viscosity solution u of the Cauchy problem

$$\begin{cases} u_t - (\frac{p}{N+p-2}) \Delta_p^1 u = 0 & \text{for } x \in \mathbb{R}^N, \quad t > 0 \\ u(\cdot, 0) = u_0 \in BUC(\mathbb{R}^N) \end{cases}$$

is given by the exponential formula

$$u(t) := \lim_{n \rightarrow \infty} \left(M_p^{t/n} \right)^n u_0 ,$$

where the statistical operator $M_p^h: BUC(\mathbb{R}^N) \rightarrow BUC(\mathbb{R}^N)$ is defined by

$$(M_p^h \varphi)(x) := (1 - q) \operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} + q \oint_{\partial B(x, \sqrt{2h})} \varphi \, ds ,$$

with $q := \frac{N(p-1)}{N+p-2}$, when $1 \leq p \leq 2$ and by

$$(M_p^h \varphi)(x) := (1 - q) \operatorname{midrange}_{\partial B(x, \sqrt{2h})} \{ \varphi \} + q \oint_{\partial B(x, \sqrt{2h})} \varphi \, ds ,$$

with $q = \frac{N}{N+p-2}$, when $p \geq 2$. Possible extensions to problems with Dirichlet boundary conditions and to homogeneous diffusion on metric measure spaces are mentioned briefly.

1. INTRODUCTION

1.1. Linear diffusion and averaging. Linear diffusion is the archetypal averaging process, and the most useful representation formulas for the solution of the heat equation convey its underlying statistics clearly. The most famous such formula involves convolution with the heat kernel: given an admissible initial value u_0 , the solution $u(x, t)$ of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{for } x \in \mathbb{R}^N \quad \text{and} \quad t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N \end{cases} \quad (1)$$

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is simply given by

$$u(x, t) = \int_{\mathbb{R}^N} P_t(x - y) u_0(y) dy, \quad (2)$$

where, for $z \in \mathbb{R}^N$ and $t > 0$,

$$P_t(z) := \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|z|^2}{4t}\right) \quad (3)$$

is the Euclidean heat kernel. Formulas (2) and (3) show that $u(x, t)$ is a weighted average of the initial data, with weights determined by a Gaussian with center x and variance $2t$. The various properties of $u(x, t)$ when $t > 0$ follow directly from these formulas; it is clear, for example, that $u(x, t)$ is smooth and that $u(x, t)$ is positive everywhere as long as u_0 is nonnegative and positive on a set of positive measure.

Heat kernel methods work beautifully in many different settings, and their development and deployment over the last thirty years have been particularly impressive (cf. [15], [47]). Heat kernel techniques are fundamentally linear, however, and therefore do not apply to parabolic equations involving nonlinear operators. Semigroup methods, on the other hand, circumvent this dependence on linearity through their reliance on resolvents ([10], [14], [42]). Following this approach, the solution of (1) is given instead by the exponential formula

$$u(t) = \lim_{n \rightarrow \infty} \left(I - \frac{t}{n} \Delta \right)^{-n} u_0. \quad (4)$$

Despite the elegance of this formula and the theory behind it, (4) does little to explain what actually happens to u_0 as it evolves according to (1); the infinite propagation speed mentioned above is certainly not obvious from (4), for example. Even worse, from the point of view of the present paper, is that verifying (4) requires first analyzing the elliptic problems

$$v - \lambda \Delta v = f, \quad \text{for given } \lambda > 0 \text{ and } f,$$

and obtaining careful estimates in appropriate spaces. Our philosophy is that we should proceed the other way: we should first obtain a practical formula for the solution of a parabolic initial-value problem, and we should then use that formula to gain insight into both the parabolic problem and the elliptic problems related to it.

Guided by this principle, this paper develops exponential formulas based not on resolvents, but on the local spatial statistics of the generator of the semigroup governing the evolution of the initial value u_0 . In

the case of the heat equation, for instance, we can exploit the relationship given in Lemma 1.1 between the Laplacian and linear averaging, in which we use the standard notation for integral average: for a measure μ , a μ -measurable set E , and a μ -measurable function f ,

$$\oint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu .$$

We also adopt the usual notation $B(x, r)$ for an open ball with center x , radius $r \geq 0$, and spherical boundary $\partial B(x, r)$.

Lemma 1.1. *For an open set $\Omega \subset \mathbb{R}^N$, $x \in \Omega$, and a smooth function $\varphi: \Omega \rightarrow \mathbb{R}$,*

$$\varphi(x) - \oint_{\partial B(x, \sqrt{2h})} \varphi(s) ds = -\frac{h}{N} \Delta \varphi(x) + o(h) . \quad (5)$$

This lemma follows directly from an elementary Taylor expansion, and identity (5) is precisely the semigroup generation formula that we need. Combining it with a classical result like the Lax Equivalence Theorem [31] or Chernoff's Product Formula [14] yields an averaging representation for the solution u of

$$\begin{cases} u_t - \frac{1}{N} \Delta u = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N . \end{cases} \quad (6)$$

Specifically, let $BUC(\mathbb{R}^N)$ denote the space of bounded, uniformly continuous functions on \mathbb{R}^N and, for a given $h > 0$, define the linear operator $M_2^h: BUC(\mathbb{R}^N) \rightarrow BUC(\mathbb{R}^N)$ by

$$(M_2^h \varphi)(x) := \oint_{\partial B(x, \sqrt{2h})} \varphi ds , \quad \text{for } x \in \mathbb{R}^N .$$

We then have the following:

Theorem 1.2. *Let $u_0 \in BUC(\mathbb{R}^N)$ be given. The function $u: [0, \infty) \rightarrow BUC(\mathbb{R}^N)$ defined by*

$$u(t) := \lim_{n \rightarrow \infty} \left(M_2^{t/n} \right)^n u_0 \quad (7)$$

is the unique bounded, continuous solution of (6).

As with the heat kernel representation (2), we can easily deduce the properties of the solution of (6) from formula (7). The regularity of u follows easily from (7), for example, thanks to the well-known smoothing effect of linear averaging. The infinite speed of propagation for the heat equation also follows easily: if u_0 is nonnegative and compactly supported, then the support of u_0 will expand a distance $\sqrt{2t/n}$ in all

directions after an application of $M_2^{t/n}$, resulting in an expansion of $\sqrt{2nt}$ in all directions after n iterations. Letting $n \rightarrow \infty$, it is clear that $u(x, t)$ will be supported on all of \mathbb{R}^N for any $t > 0$. For future reference, note that this same calculation establishes an infinite speed of propagation for the Cauchy problems considered in Section 4.

When $N = 1$, the operator M_2^h is particularly simple, since we then have

$$(M_2^h \varphi)(x) = \frac{\varphi(x - \sqrt{2h}) + \varphi(x + \sqrt{2h})}{2}. \quad (8)$$

In this case, formula (7) is a fully discrete forward Euler scheme for the heat equation in one space dimension. Well-known arguments related to simple random walks on the line [30] then lead directly from (7) to the heat kernel representation (2). In higher dimensions, the averaging operator M_2^h no longer acts on discrete sets, but we can still combine the obvious discrete approximation of the average over a sphere with standard random walk techniques to derive (2) from (7).

The exponential formula (7) is thus clearly related to finite difference methods for the heat equation, but it does not seem to have been used as a purely theoretical tool, despite its correspondence with our common intuition about linear diffusion. Furthermore, since our interest in (7) is not driven by numerics, the computational limitations of forward Euler schemes are irrelevant here. In fact, we contend that formula (7) is superior to other representations of the solution of (6); formula (7) exposes the averaging inherent in linear diffusion, permits an elementary analysis of solutions of (6), can be modified easily to accommodate Dirichlet boundary conditions (see Section 5 below), leads naturally to the mean value property of harmonic functions, and can be used to derive the heat kernel formula (2). Even more significant is the fact that straightforward variations of formula (7) provide representations of the solutions of statistically-driven nonlinear diffusion equations. Substantiating this last claim is the goal of this paper.

1.2. Overview of paper and background results. As intimated above, this paper develops exponential formulas for the continuous viscosity solutions of Cauchy problems of the general form

$$\begin{cases} u_t + F(Du, D^2u) = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \quad (9)$$

where $u_0 \in BUC(\mathbb{R}^N)$, Du denotes the spatial gradient of the real-valued function u , D^2u denotes its spatial Hessian, and $F = F(p, X)$

satisfies the conditions listed below; as usual, S^N denotes the space of $N \times N$ real symmetric matrices with its standard partial ordering.

- (i) $F : \mathbb{R}^N \setminus \{0\} \times S^N \rightarrow \mathbb{R}$ is continuous; excluding the case $p = 0$ accommodates singular gradient dependence in (9).
- (ii) F is 1-homogeneous: given $p \in \mathbb{R}^N \setminus \{0\}$, $X \in S^N$, and $\alpha \in \mathbb{R}$,

$$F(\alpha p, \alpha X) = \alpha F(p, X).$$

- (iii) F is degenerate elliptic: for a given $p \in \mathbb{R}^N \setminus \{0\}$,

$$F(p, X) \leq F(p, Y) \quad \text{when} \quad Y \leq X.$$

Since representation formulas for solutions are the focus of this paper, we presume a familiarity with viscosity solutions and refer to other works for definitions, existence proofs, comparison principles, and so on. The Users' Guide [9] and the lecture notes [5] are basic references, several papers by Juutinen, Kawohl and their coauthors ([2], [21], [24], [23], [26], [27]) address important issues related to comparison principles and the definitions of solutions, and the fundamental paper [13] by Giga, Goto, Ishii and Sato is an essential reference (see also [12]), as it contains the comparison principle on which our uniqueness statements depend. Also, the results that follow all depend on the modern viscosity version of Chernoff's Product Formula developed by Barles and Souganidis in [7].

To summarize what follows, Section 2 shows that replacing the linear average in (7) with the median yields an exponential formula for motion by mean curvature; as discussed there, the median is a nonlinear average that must be handled a bit carefully. It is easier to work with the midrange, which averages the infimum and supremum of a function over a set, and Section 3 proves that iterating this particular nonlinear average provides a formula for the solution of the Cauchy problem for the parabolic infinity-Laplacian. Section 4 combines the results from Sections 1, 2 and 3 to establish exponential formulas for the Cauchy problems involving the parabolic 1-homogeneous p -Laplacian for $1 \leq p \leq \infty$. (The 1-homogeneous p -Laplacian is also known as the normalized or game-theoretic p -Laplacian.) Finally, Section 5 speculates on possible generalizations of our results to parabolic problems with Dirichlet boundary conditions and to homogeneous diffusion on metric measure spaces.

All of the exponential formulas proven below are the same when $N = 1$, since the averaging operators used all reduce to (8) in that case. We therefore assume henceforth that $N \geq 2$.

2. MEDIAN AND MEAN CURVATURE FLOW IN \mathbb{R}^N

This section establishes a statistical exponential formula for the solution u of the level set formulation of mean curvature flow,

$$\begin{cases} u_t - |Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0, & \text{for } x \in \mathbb{R}^N, \ t > 0, \\ u(x, 0) = u_0(x), & \text{for } x \in \mathbb{R}^N. \end{cases} \quad (10)$$

This well-known front propagation model was introduced by Osher and Sethian [41] and has since been studied extensively by many authors. Our exponential formula for its solution is a simple nonlinear analogue of the averaging representation (7) for the solution u of the heat equation (6); to provide a context for the formula, we briefly review some particularly relevant earlier work.

Since (10) appeared in [41], developing efficient and provably convergent algorithms for approximating its solution has been a fundamental problem. In [38], Bence, Merriman and Osher proposed their famous algorithm for mean curvature motion; during each iteration, their algorithm solves the heat equation with appropriate initial data and then thresholds the resulting solution. Their algorithm's performance motivated much work on its convergence properties, resulting in the basic references [6], [11], and [20]. Of these three papers, the work by Evans [11] is perhaps most closely related to the present paper, as it applies the resolvent-based nonlinear semigroup machinery of Crandall and Liggett [10] to prove convergence of the Bence-Merriman-Osher scheme.

Shortly thereafter, Catté, Dibos and Koepfler [8] established a different Crandall-Liggett exponential formula for the solution of (10) when $N = 2$, basing their development on an axiomatic approach to image processing [3]. Specifically, Catté et al. proved that the solution u of (10) has the representation

$$u(t) = \lim_{n \rightarrow \infty} C_{t/n}^n u_0, \quad (11)$$

where

$$\begin{aligned} C_h \varphi &:= \frac{1}{2} \{ S_{2h} \varphi + I_{2h} \varphi \}, \\ (I_h \varphi)(x) &:= \inf_{\theta \in [0, \pi)} \sup_{x + \sigma(\theta, h)} \{ \varphi(y) \}, \\ (S_h \varphi)(x) &:= \sup_{\theta \in [0, \pi)} \inf_{x + \sigma(\theta, h)} \{ \varphi(y) \}, \end{aligned}$$

and $\sigma(\theta, h)$ denotes a segment centered at 0 with direction θ and length $2\sqrt{2h}$.

Although [8] appeared roughly twenty years ago, it has become better known through a more recent work by Kohn and Serfaty [29] that produced a variant of formula (11) from a very different perspective. Kohn and Serfaty arrived at their version of (11) by way of the dynamic programming principle for a simple deterministic two-player game in \mathbb{R}^2 that we review briefly. During each round of the game, the first player chooses a direction $v \in S^1$ that the second player either accepts or reverses, thereby determining the direction $w = \pm v$; the game position then moves a distance $\sqrt{2h}$ in the direction w . When the game ends, the first player pays the second player the amount $u_0(x_T)$, where x_T is the game position when the game ends at time T . Since the first player's value function u_h corresponds to playing optimally, u_h satisfies the dynamic programming principle

$$u_h(x, kh) = \min_{v \in S^1} \max_{b=\pm 1} \left\{ u_h(x + \sqrt{2h}bv, (k+1)h) \right\}, \quad (12)$$

with $u_h(x, T) = u_0(x)$. Using (12), Kohn and Serfaty proved that the value functions u_h converge, as $h \rightarrow 0$, to the solution of (10) (after a simple change of variables to go from this terminal-value game to the initial-value formulation above). Equation (12) and the results of [29] therefore show that we can approximate the mean curvature motion of a curve, over a small time step h , by tracking the midpoint of a segment of length $2\sqrt{2h}$ as its endpoints traverse the curve; iterating this procedure yields an exponential formula for the solution of (10).

Ruuth and Merriman presented a similar description of mean curvature flow, from yet another viewpoint, in [44], an interesting paper that does not seem to have received sufficient attention. They proved that, for a given closed curve γ in the plane, one can approximate the mean curvature flow of γ for a small time step h by tracking the center of a circle of radius $\sqrt{2h}$ as it traverses γ in such a way that exactly half of its area is always inside γ . Note that the approximation scheme corresponding to (12), on the other hand, tracks the center of a circle as it traverses γ in such a way that half of its circumference is always inside γ .

While the paper by Kohn and Serfaty was making the rounds as a preprint, Oberman published his paper on provably convergent median schemes for mean curvature flow [39]. In it, he presented a forward Euler method for approximating the solution of (10) on a rectangular grid; at each iteration, the algorithm updates each grid value with the median of its neighboring grid values, with neighbors defined carefully so as to reduce errors from poor angular resolution. In particular, using a small grid spacing and a wide computational stencil enables

the selection of neighbors that approximate a circle centered at the grid point of interest.

All of these papers have suggested connections between mean curvature flow and appropriately interpreted median operators, with Oberman's work doing so explicitly in a discrete context. To develop these connections further, we summarize the properties of medians of measurable and continuous functions, beginning with the definition and proceeding to some recent results that should be of independent interest.

Definition 2.1 ([48]). If $u: E \rightarrow \mathbb{R}$ is measurable and $0 < |E| < \infty$, then m is a *median* of u over E if and only if

$$|\{u < m\}| \leq \frac{1}{2}|E| \quad \text{and} \quad |\{u > m\}| \leq \frac{1}{2}|E|.$$

We denote the set of all medians of u over E by $\text{median}_E\{u\}$. As discussed in [48], $\text{median}_E\{u\}$ is a non-empty compact interval, and

$$\text{median}_E\{\alpha u + \beta\} = \alpha \text{median}_E\{u\} + \beta \quad (13)$$

for any constants α and β . We will exploit these homogeneity and translation invariance properties below, as well as the obvious stability of medians: if $u \in L^\infty(E)$, then $m \leq \|u\|_\infty$ for any $m \in \text{median}_E\{u\}$.

If u is merely measurable, the set of medians of u over E can clearly have positive Lebesgue measure. Continuous functions, on the other hand, have unique medians over compact connected sets, as shown by the following results from [17]. These facts pave the way for further analysis and explain our restriction to continuous initial data throughout this paper. In the statement of Lemma 2.2, $LSC(E)$, $USC(E)$ and $C(E)$ denote, respectively, the lower semicontinuous, upper semicontinuous, and continuous real-valued functions on the measurable set E .

Lemma 2.2 ([17]). *Suppose that $L \in LSC(E)$, $U \in USC(E)$, $L \geq U$, and that $E \subset \mathbb{R}^k$ is compact and connected. If $m \in \text{median}_E L$ and $M \in \text{median}_E U$, then $m \geq M$.*

Corollary 2.3 ([17]). *Suppose that $E \subset \mathbb{R}^k$ is compact and connected. If $v \in C(E)$, then the median of v over E is unique. Moreover, if $u \in C(E)$ satisfies $u \geq v$, then $\text{median}_E u \geq \text{median}_E v$.*

Lemma 1.1 is the essential ingredient in the proof of the exponential formula (7) for the solution of (6); it shows that repeated linear averaging provides a consistent approximation scheme for linear diffusion.

The next result shows that the median, a nonlinear average, plays a corresponding role for mean curvature flow. It is new in dimensions $N > 2$, having been established in [16] for functions of two variables. Henceforth, it will be convenient to let Δ_1 denote the 1-Laplacian operator, the elliptic part of equation (10); Δ_1 is thus defined formally by

$$\Delta_1 \varphi := |D\varphi| \operatorname{div} \left(\frac{D\varphi}{|D\varphi|} \right). \quad (14)$$

It will also be useful below to recall that, when $D\varphi(x) \neq 0$,

$$\Delta_1 \varphi(x) = \operatorname{tr} (D^2 \varphi(x)|_{\Sigma}) , \quad (15)$$

where Σ is the plane orthogonal to $D\varphi(x)$; this description of the 1-Laplacian plays a key role in the related papers [25] and [28].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ be open, $N \geq 2$. For $x \in \Omega$ and a smooth function $\varphi: \Omega \rightarrow \mathbb{R}$ with $|D\varphi(x)| \neq 0$,*

$$\varphi(x) - \operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = - \frac{h}{N-1} \Delta_1 \varphi(x) + o(h). \quad (16)$$

Proof. We begin by recalling the proof given in [16] when $N = 2$. Since $|D\varphi(x)| \neq 0$, the Implicit Function Theorem guarantees that, for sufficiently small $h > 0$, the level sets of φ are smooth curves that foliate the closure of the ball $B(x, \sqrt{2h})$. There is a unique level curve corresponding to the median of φ over the circle $\partial B(x, \sqrt{2h})$; by definition of the median, this curve must separate the circle into two arcs of equal length, yielding antipodal points $y_h^+, y_h^- \in \partial B(x, \sqrt{2h})$ such that

$$\varphi(y_h^+) = \varphi(y_h^-) = \operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \}. \quad (17)$$

Letting

$$v_h := \frac{y_h^+ - x}{\sqrt{2h}} = - \frac{y_h^- - x}{\sqrt{2h}},$$

we have the Taylor expansions

$$\varphi(y_h^+) = \varphi(x) + \sqrt{2h} D\varphi(x) \cdot v_h + h D^2 \varphi(x) v_h \cdot v_h + o(h)$$

and

$$\varphi(y_h^-) = \varphi(x) - \sqrt{2h} D\varphi(x) \cdot v_h + h D^2 \varphi(x) v_h \cdot v_h + o(h),$$

which must be equal by (17). Subtracting one from the other, we see that

$$2\sqrt{2h} D\varphi(x) \cdot v_h = o(h),$$

from which it follows that

$$v_h = \frac{D\varphi(x)^\perp}{|D\varphi(x)|} + e_h, \quad (18)$$

where

$$D\varphi(x)^\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} D\varphi(x) \quad \text{and} \quad 2\sqrt{2h} D\varphi(x) \cdot e_h = o(h).$$

Substituting the expression for v_h from (18) into either Taylor expansion and calculating directly completes the proof in this case.

Before proceeding to the proof in higher dimensions, we rephrase the proof just given in a form that will be easier to generalize. Using a normal coordinate system (y_1, y_2) centered at x , we can assume that $x = (0, 0)$ and that $D\varphi(x)$ determines the direction of the y_2 -axis. Up to second order (i.e., modulo $o(h)$ errors since the radius is $\sqrt{2h}$), we have the following: the level curve passing through x is the parabola $y_2 = -\frac{1}{2}\kappa y_1^2$ (where κ is the curvature at the vertex x), the level curves of φ in $B(x, \sqrt{2h})$ are translates of this parabola, and the median corresponds to the level curve $y_2 = \kappa h - \frac{1}{2}\kappa y_1^2$. In this coordinate system, then,

$$\varphi(\pm\sqrt{2h}, 0) = \operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} + o(h),$$

after which the rest of the proof remains the same.

When $N = 3$, we know by the Implicit Function Theorem that, for sufficiently small $h > 0$, the level sets of φ are smooth 2-dimensional surfaces that foliate the closure of the ball $B(x, \sqrt{2h})$. Introduce coordinates (y_1, y_2, y_3) such that $x = (0, 0, 0)$ and the y_3 -axis is parallel to $D\varphi(x)$; up to second order, the level sets of φ in $B(x, \sqrt{2h})$ are translates of a quadratic polynomial $p(y_1, y_2)$ (namely, the second fundamental form of the level surface through x), and the median corresponds to the level surface $y_3 = \alpha + p(y_1, y_2)$ whose intersection Γ with $\partial B(x, \sqrt{2h})$ bisects the surface area of $\partial B(x, \sqrt{2h})$. Consequently, since $p(y_1, y_2) = p(-y_1, -y_2)$, Γ must intersect the plane $\{y_3 = 0\}$ in at least two distinct pairs of antipodal points. These pairs of antipodal points yield independent unit vectors v_h and w_h such that

$$\operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = \varphi(x) \pm \sqrt{2h} D\varphi(x) \cdot v_h + h D^2\varphi(x) v_h \cdot v_h + o(h)$$

and

$$\operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = \varphi(x) \pm \sqrt{2h} D\varphi(x) \cdot w_h + h D^2\varphi(x) w_h \cdot w_h + o(h),$$

from which we conclude as above that v_h and w_h are nearly orthogonal to $D\varphi(x)$. More precisely, we have

$$\operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = \varphi(x) + h D^2 \varphi(x) v_h \cdot v_h + o(h) \quad (19)$$

and

$$\operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = \varphi(x) + h D^2 \varphi(x) w_h \cdot w_h + o(h) , \quad (20)$$

and averaging equations (19) and (20) yields

$$\operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = \varphi(x) + \frac{h}{2} (D^2 \varphi(x) v_h \cdot v_h + D^2 \varphi(x) w_h \cdot w_h) + o(h). \quad (21)$$

Using the characterization (15) of the 1-Laplacian above, we can rewrite (21) in the form

$$\operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = \varphi(x) + \frac{h}{2} \Delta_1 \varphi(x) + o(h) ,$$

which is equation (16) when $N = 3$.

A similar argument now applies in higher dimensions, using a quadratic polynomial of $(N - 1)$ variables to approximate the smooth submanifold on which φ achieves its median.

□

Given $h > 0$ and $\varphi \in BUC(\mathbb{R}^N)$, we define the nonlinear averaging operator $M_1^h: BUC(\mathbb{R}^N) \rightarrow BUC(\mathbb{R}^N)$ by

$$(M_1^h \varphi)(x) := \operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} , \quad \text{for } x \in \mathbb{R}^N . \quad (22)$$

The fact that $M_1^h \varphi$ is bounded when φ is bounded follows directly from the stability of the median mentioned above, and Proposition 2.2 of [17] verifies that $M_1^h \varphi$ is uniformly continuous whenever φ is uniformly continuous. We see from the preceding results that the operator M_1^h has the following important properties:

- (i) translation invariance: $M_1^h(v + c) = M_1^h v + c$ for any $v \in BUC(\mathbb{R}^N)$ and $c \in \mathbb{R}$.
- (ii) monotonicity: $M_1^h v \leq M_1^h w$ whenever $v, w \in BUC(\mathbb{R}^N)$ satisfy $v \leq w$.
- (iii) 1-homogeneity: if $\alpha \in \mathbb{R}$ and $v \in BUC(\mathbb{R}^N)$, then $M_1^h(\alpha v) = \alpha M_1^h(v)$.
- (iv) stability: $\|M_1^h v\|_\infty \leq \|v\|_\infty$ for any $v \in BUC(\mathbb{R}^N)$.

(v) consistency: for any smooth φ with nonvanishing gradient,

$$\lim_{h \rightarrow 0} \left(\frac{\varphi - (M_1^h \varphi)}{h} \right) = - \frac{1}{N-1} \Delta_1 \varphi .$$

Consequently, a direct application of the framework developed in [7] proves the following:

Theorem 2.5. *Let $u_0 \in BUC(\mathbb{R}^N)$ be given. The function $u: [0, \infty) \rightarrow BUC(\mathbb{R}^N)$ defined by*

$$u(t) := \lim_{n \rightarrow \infty} \left(M_1^{t/n} \right)^n u_0 \quad (23)$$

is the unique viscosity solution of

$$\begin{cases} u_t - \frac{1}{N-1} \Delta_1 u = 0 & \text{for } x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \quad (24)$$

As mentioned toward the end of Section 1, the uniqueness of this solution follows from the relevant comparison principle ([12], [13]).

Although we only stated the exponential formula (7) for the solution of the heat equation in terms of averages over spheres, an analogous formula based on averages over balls certainly holds. It is not clear whether a version of Theorem 2.5 based on medians over closed balls is true, as we do not have a proof of the consistency result in Lemma 2.4 when we replace spheres with closed balls. The earlier results of Bence-Merriman-Osher [38] and Ruuth-Merriman [44], however, suggest that using medians over closed balls should work; this would be an interesting issue to resolve.

3. MIDRANGES AND THE PARABOLIC ∞ -LAPLACIAN

We now consider the Cauchy problem

$$\begin{cases} u_t - \Delta_\infty u = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \quad (25)$$

where $u_0 \in BUC(\mathbb{R}^N)$ and the ∞ -Laplacian Δ_∞ is defined by

$$\Delta_\infty \varphi := \frac{1}{|D\varphi|^2} \sum_{i,j=1}^N \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

for smooth φ with $|D\varphi| \neq 0$. In contrast to the heat equation and the mean curvature equation, there are surprisingly few papers devoted to parabolic problems involving the ∞ -Laplacian. In [23], Juutinen

and Kawohl answered basic existence and uniqueness questions for the Cauchy problem (25) as well as its analogue on bounded domains with Dirichlet boundary conditions; they proved, in particular, that (25) has a unique bounded solution. Subsequently, Akagi, Juutinen and Kajikiya studied the asymptotic behavior of the solutions of these evolution problems, proving, among other results, that the optimal decay rate (in $L^\infty(\mathbb{R}^N)$) of the solution of (25) when u_0 has compact support is $(t+1)^{-1/6}$. More recently, Manfredi, Parviainen and Rossi [36] proved an asymptotic statistical characterization of solutions of (25) that complements the present work.

To establish a statistical exponential formula for the solution of (25), we recall the following relationship between the ∞ -Laplacian and the average of extreme values:

Lemma 3.1. *For an open set $\Omega \subset \mathbb{R}^N$, $x \in \Omega$, and a smooth function $\varphi: \Omega \rightarrow \mathbb{R}$ with $|D\varphi(x)| \neq 0$,*

$$\varphi(x) - \operatorname{midrange}_{\partial B(x, \sqrt{2h})} \{ \varphi \} = -h \Delta_\infty \varphi(x) + o(h), \quad (26)$$

where, for any compact set $K \subset \mathbb{R}^N$,

$$\operatorname{midrange}_K \{ \varphi \} := \frac{1}{2} \left(\max_K \{ \varphi \} + \min_K \{ \varphi \} \right).$$

Identity (26) has been used in various forms elsewhere ([28], [33], [32], [36], [40]) and follows from the elementary fact that the gradient is the direction of steepest ascent. Note that, like the median operator discussed in the previous section, the midrange operator computes a nonlinear average and is monotone, stable, translation invariant (in the sense used earlier), and 1-homogeneous. Moreover, Lemma 3.1 shows that the midrange operator provides a consistent approximation of the ∞ -Laplacian. Proceeding as in Section 2, we therefore define the nonlinear averaging operator $M_\infty^h: BUC(\mathbb{R}^N) \rightarrow BUC(\mathbb{R}^N)$ by

$$(M_\infty^h \varphi)(x) := \operatorname{midrange}_{\partial B(x, \sqrt{2h})} \{ \varphi \}, \quad \text{for } x \in \mathbb{R}^N, \quad (27)$$

and apply the machinery of [7] to obtain

Theorem 3.2. *Let $u_0 \in BUC(\mathbb{R}^N)$ be given. The function $u: [0, \infty) \rightarrow BUC(\mathbb{R}^N)$ defined by*

$$u(t) := \lim_{n \rightarrow \infty} (M_\infty^{t/n})^n u_0 \quad (28)$$

is the unique bounded, continuous viscosity solution of (25).

4. HOMOGENEOUS DIFFUSION IN \mathbb{R}^N

The evolution equations studied in Sections 1, 2 and 3 belong to the one-parameter family of Cauchy problems

$$\begin{cases} u_t - c_{p,N} \Delta_p^1 u = 0 & \text{for } x \in \mathbb{R}^N \text{ and } t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \quad (29)$$

where

$$c_{p,N} := \frac{p}{N + p - 2}$$

and the 1-homogeneous p -Laplacian Δ_p^1 is defined, for $1 \leq p \leq \infty$, by

$$\Delta_p^1 \varphi := \begin{cases} (1 - \frac{1}{p}) \Delta \varphi + (\frac{2}{p} - 1) \Delta_1 \varphi & \text{if } 1 \leq p \leq 2, \\ \frac{1}{p} \Delta \varphi + (1 - \frac{2}{p}) \Delta_\infty \varphi & \text{if } p \geq 2. \end{cases} \quad (30)$$

Note that $\Delta_1^1 = \Delta_1$ and $\Delta_\infty^1 = \Delta_\infty$, while $\Delta_2^1 = \frac{1}{2} \Delta$.

Given $h > 0$ and $p \in [1, \infty]$, we define the statistical operator $M_p^h: BUC(\mathbb{R}^N) \rightarrow BUC(\mathbb{R}^N)$ by

$$(M_p^h \varphi)(x) := \begin{cases} (1 - q) \operatorname{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} + q \int_{\partial B(x, \sqrt{2h})} \varphi ds, & 1 \leq p \leq 2, \\ (1 - q) \operatorname{midrange}_{\partial B(x, \sqrt{2h})} \{ \varphi \} + q \int_{\partial B(x, \sqrt{2h})} \varphi ds, & p \geq 2, \end{cases}$$

where

$$q = q(p, N) := \begin{cases} \frac{N(p-1)}{N+p-2}, & \text{if } 1 \leq p \leq 2, \\ \frac{N}{N+p-2}, & \text{if } p \geq 2. \end{cases} \quad (31)$$

Since the operator M_p^h is a simple linear combination of the averaging operators studied earlier, it clearly enjoys the same properties of homogeneity, stability, monotonicity, and translation invariance, with the consistency condition

$$\lim_{h \rightarrow 0} \left(\frac{\varphi - (M_p^h \varphi)}{h} \right) = -c_{p,N} \Delta_p^1 \varphi \quad (32)$$

following from Lemmas 1.1, 2.4, and 3.1. As in Sections 2 and 3, we can therefore combine the results of [7] and the comparison principle from [13] to establish

Theorem 4.1. *Let $u_0 \in BUC(\mathbb{R}^N)$ be given. The function $u: [0, \infty) \rightarrow BUC(\mathbb{R}^N)$ defined by*

$$u(t) := \lim_{n \rightarrow \infty} (M_p^{t/n})^n u_0 \quad (33)$$

is the unique bounded, continuous viscosity solution of (29).

As remarked after the statement of Theorem 1.2, the infinite speed of propagation of the heat equation follows directly from formula (7). The exact same argument shows that problem (29) also exhibits an infinite speed of propagation as long as the coefficient q is nonzero in the definition of M_p^h ; $q \neq 0$ precisely when $1 < p < \infty$. When $p = 1$, the solution of the mean curvature equation (10) corresponding to a nonnegative compactly supported initial value will vanish in finite time, a property known as finite extinction; this fact can be verified from formula (23) by analyzing what happens with each application of the median operator M_1^h . When $p = \infty$, the infinite speed of propagation of equation (25) follows from formula (28) since each application of the midrange operator M_∞^h will enlarge the support of a compactly supported function by an amount $\sqrt{2h}$ in the direction of steepest descent; this reflects the one-dimensional (and thus highly degenerate) nature of the infinity-Laplacian, something that is carefully explored and exploited in [1].

The relationship between the 1-homogeneous p -Laplacian Δ_p^1 (also known as the normalized or game-theoretic p -Laplacian, cf. [25], [36], [43]) and the classical p -Laplacian is fairly well-known by now, but we review it briefly for completeness. For $p \in (1, \infty)$, the classical p -Laplacian Δ_p is the operator defined formally by

$$\Delta_p \varphi := \operatorname{div} (|D\varphi|^{p-2} D\varphi) .$$

This operator arises naturally in variational problems ([18], [35]), but more recent work ([21], [24]) has shown that it is also amenable to viscosity methods. With such methods in mind, we define a p -harmonic function to be a continuous viscosity solution u of

$$-\Delta_p u = 0 . \quad (34)$$

Formal calculations show that

$$\Delta_p u = |Du|^{p-2} (\Delta u + (p-2)\Delta_\infty u) , \quad (35)$$

an identity used in [24] to prove that u is p -harmonic if and only if

$$-\Delta u - (p-2)\Delta_\infty u = 0$$

in the viscosity sense. We also find that

$$\Delta_1 u = \Delta u - \Delta_\infty u ,$$

from which we have the important alternative decomposition

$$\Delta_p u = |Du|^{p-2} ((p-1)\Delta u + (2-p)\Delta_1 u) . \quad (36)$$

Based on the work in [21] and [24], it follows that u is p -harmonic if and only if u is a viscosity solution of

$$(p-1)\Delta u + (2-p)\Delta_1 u = 0 . \quad (37)$$

These results motivate the definition of the 1-homogeneous p -Laplacian given above, and we see that Δ_p^1 excludes the gradient-dependent factor that makes the classical p -Laplacian $(p-1)$ -homogeneous. Thanks to this difference in homogeneity, the elliptic and parabolic problems related to the 1-homogeneous p -Laplacian seem easier to analyze than the corresponding problems for the classical p -Laplacian.

5. DIRECTIONS FOR FUTURE WORK

There are many open problems related to the exponential formulas proven earlier. Of these, this final section focuses on two particular directions in which we are working to develop these ideas further.

5.1. Dirichlet boundary conditions. By considering Cauchy problems exclusively so far, we have avoided complications caused by boundaries and boundary conditions. Dirichlet boundary conditions, in particular, pose significant challenges for nonlinear parabolic problems; at the very least, one expects some geometric restrictions on the boundary of the domain to be necessary in order to obtain a solution that attains the prescribed boundary values continuously.

In an attempt to attack such problems, we recall the approach we applied to stationary Dirichlet problems in [17]: given $h > 0$, $p \in [1, \infty]$, and a domain $\Omega \subset \mathbb{R}^N$ with nonempty boundary $\partial\Omega$, define the statistical operator $M_p^h: BUC(\overline{\Omega}) \rightarrow BUC(\overline{\Omega})$ by

$$(M_p^h \varphi)(x) := \varphi(x) \quad \text{for } x \in \partial\Omega$$

and

$$(M_p^h \varphi)(x) := \begin{cases} (1-q) \operatorname{median}_{\partial B_x^h} \{ \varphi \} + q \int_{\partial B_x^h} \varphi dy , & 1 \leq p \leq 2 \\ (1-q) \operatorname{midrange}_{\partial B_x^h} \{ \varphi \} + q \int_{\partial B_x^h} \varphi dy , & p \geq 2 \end{cases}$$

for $x \in \Omega$, where $q = q(p, N)$ is defined by (31) and the open balls B_x^h are defined by

$$B_x^h := B(x, r^h(x)) ,$$

with

$$r^h(x) := \begin{cases} \sqrt{2h} & \text{if } \text{dist}(x, \partial\Omega) \geq \sqrt{2h}, \\ \text{dist}(x, \partial\Omega) & \text{otherwise.} \end{cases}$$

Note that this reduces to the definition of M_p^h in Section 4 if the boundary of Ω happens to be empty.

Since M_p^h is a local operator, it still satisfies the consistency condition (32), as well as the monotonicity, stability, and homogeneity properties that we have been using. We would therefore like to invoke the results of [7] yet again to conclude that

$$u(t) := \lim_{n \rightarrow \infty} (M_p^{t/n})^n u_0 \quad (38)$$

is the unique bounded, continuous viscosity solution on $\overline{\Omega} \times [0, \infty)$ of

$$\begin{cases} u_t - c_{p,N} \Delta_p^1 u = 0 & \text{for } x \in \Omega \text{ and } t > 0, \\ u(x, t) = u_0(x) & \text{for } x \in \partial\Omega \text{ and } t > 0, \\ u(x, 0) = u_0(x) & \text{for } x \in \overline{\Omega}, \end{cases} \quad (39)$$

where $u_0 \in BUC(\overline{\Omega})$, but this will simply not work for arbitrary p and Ω . When Dirichlet conditions are imposed, we lack an appropriate comparison principle in general, and our understanding of the interaction of M_p^h with boundary conditions is far from complete. We handled similar issues in [17] by using stringent definitions of subsolutions and supersolutions; we then found such sub- and supersolutions by requiring $\partial\Omega$ to be strictly convex. The basic problem is to determine conditions on Ω and p that will guarantee the existence of a unique continuous viscosity solution of (39).

Such conditions are known for bounded domains and certain values of p . When $p = \infty$, for instance, Juutinen and Kawohl [23] proved that problem (39) has a unique, bounded viscosity solution for any bounded domain Ω , a surprisingly general result made possible by the extreme degeneracy of the infinity-Laplacian. When $p = 2$, problem (39) has a unique bounded solution if barriers exist at each point on $\partial\Omega$ [34]. When $p = 1$, Sternberg and Ziemer [45] proved that (39) has a unique bounded solution as long as $\partial\Omega$ is strictly convex. Under this assumption, Ilmanen, Sternberg and Ziemer [19] analyzed the asymptotic behavior of this solution, proving that the solution approaches the unique function of least gradient on Ω as $t \rightarrow \infty$ if the boundary data is C^2 ; if the boundary data is only continuous, however, then this asymptotic limit is merely 1-harmonic. Since 1-harmonic functions need not be unique, one naturally wonders which one is chosen as $t \rightarrow \infty$.

One should compare these results with Juutinen's work on limits of p -harmonic functions as $p \rightarrow 1$ [22].

What happens for other values of p needs to be investigated more thoroughly. Also, the exponential formula (38) makes sense for any $u_0 \in BUC(\overline{\Omega})$; how does the function $u(t)$ defined by (38) behave for various values of p and various domains Ω ? For example, what are the properties of this function when $p = 1$ and Ω does not have a strictly convex boundary? In this case, one expects $u(t) \in BV(\Omega)$ for $t > 0$; when and where do discontinuities develop? These are just a few of the questions that need to be explored.

5.2. Homogeneous diffusion on metric measure spaces. Instead of working on \mathbb{R}^N , suppose that we have a finite-dimensional metric space (X, d) equipped with a Radon measure μ on spheres $\partial B(x, r)$, for $x \in X$ and $r \geq 0$. Given $p \in [1, \infty]$ and $h > 0$, we can define the local statistical operator $M_p^h: BUC(X) \rightarrow BUC(X)$ as before, since

$$\text{median}_{\partial B(x, \sqrt{2h})} \{ \varphi \} , \quad \int_{\partial B(x, \sqrt{2h})} \varphi(s) d\mu , \quad \text{and} \quad \text{midrange}_{\partial B(x, \sqrt{2h})} \{ \varphi \}$$

all make sense in this setting. The flow defined by the exponential formula (33) thus also makes sense, at least *a priori*, in this setting; what are its properties? Even in the simplest case when $p = 2$ and the metric measure space (X, d, μ) is a Riemannian manifold without boundary, when is (33) a formula for the solution of the heat equation on X ? It seems reasonable to suspect that this holds whenever X is a harmonic manifold, but the answer to this elementary question does not appear to be readily available in the literature. Similarly, could (33) be used to derive the heat kernel explicitly on non-Euclidean spaces? When $p = 1$, does formula (33) provide useful insights into mean curvature flow on non-Euclidean spaces? The many questions related to these operators and this exponential formula when the underlying space is not \mathbb{R}^N are all open and intriguing.

To generalize even further, we can also consider these questions for maps from the metric measure space (X, d, μ) into a more general metric space (Y, δ) than \mathbb{R} . To that end, let $p \in [1, \infty]$ be given, and define the p -mean of the continuous function $\varphi: X \rightarrow Y$ over a compact connected set $E \subset X$ by

$$m_p(\varphi, E) := \arg \min_{m \in Y} \left\{ \int_E (\delta(\varphi(x), m))^p d\mu(x) \right\} .$$

What happens when we replace M_p^h in (33) with the nonlinear averaging operator

$$(A_p^h \varphi)(x) := m_p(\varphi, \partial B(x, \sqrt{2h})) ?$$

There should be interesting connections between this operator, the resulting exponential formula, and semigroup methods for p -harmonic mappings between metric spaces (cf. [4], [37], [46]).

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